Linear Equations Solutions
Presented by QSC

Graphing

A line that goes upward from left to right must have positive slope.

A line that goes downward from left to right must have negative slope.

Recall the equation for slope is: $m = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

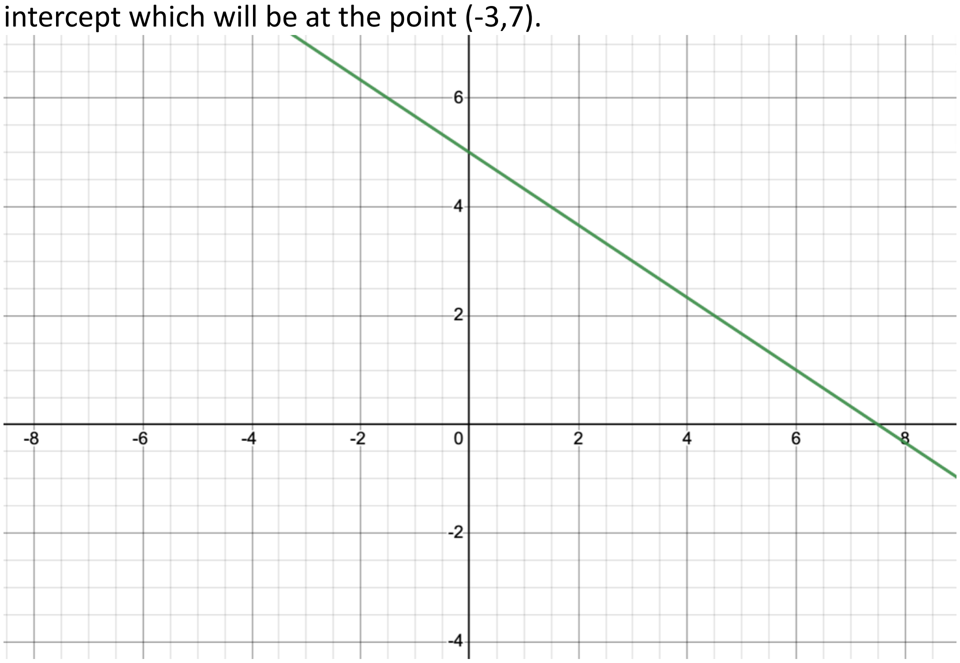
Two methods for graphing a line are
Method 1: Plotting points.

Method 2: Determining the slope and y-intercept based on the equation of the line.

Let's graph the following lines.

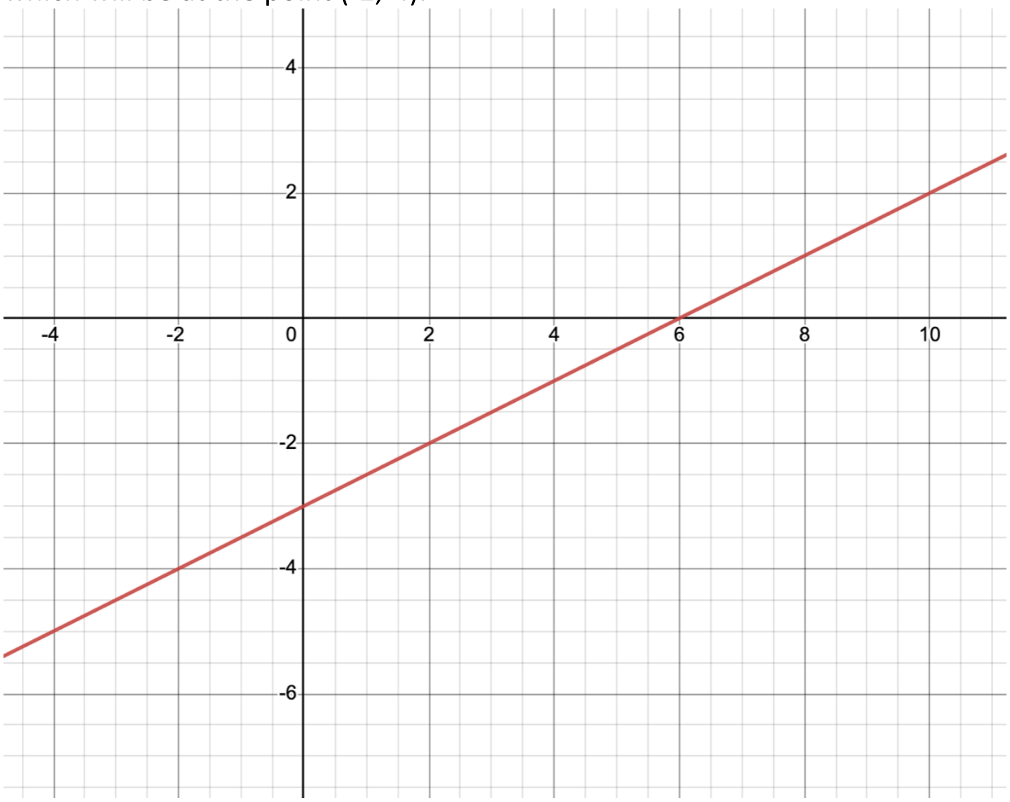
- Graph $y = -\frac{2}{3}x + 5$

Solution: The graph will be a line that goes downward from left to right. We can start from the y-intercept (0,5) and go down vertically by two, and to the right by 3 which will be at the point (3, 3). We could also plot another point by going up by two, and to the left by 3 from the y-intercept which will be at the point (-3,7).



- Graph $y = \frac{1}{2}x - 3$

Solution: The graph will be a line that goes upward from left to right. We can start from the y-intercept (0,-3) and up vertically by 1, and to the right by 2 which will be the point (2, -2). We could also plot another point by going down by 1, and to the left by 2 from the y-intercept which will be at the point (-2,-4).



<u>Equations</u>

- Slope-intercept form:
    - Write the equation: $y = mx + b$
        - $m = slope$
        - b is the <u>y-</u>value of the <u>y-intercept</u> on the line

- Point-slope form
    - Write the equation: $y - y_1 = m(x - x_1)$
        - $m = slope$
        - (x_1, y_1) is a <u>point</u> on the line
        - We can manipulate point-slope form into slope-intercept form (and vice versa)

- General form
    - Write the equation: $Ax + By = C$
        - *A, B, C are real numbers*
        - Observe that we can manipulate the general form into other forms of lines as well

- Example 1: Write the equation of the line that has slope of $\frac{3}{5}$ and y-intercept $(0,2)$.

Solution: We can use slope-intercept form, $y = mx + b$, since the two pieces of information are given. $m = \frac{3}{5}, b = 2$. So the equation of the line will be $y = \frac{3}{5}x + 2$.

- Example 2: Write the equation of a line that has slope $-\frac{2}{7}$ and goes through the point $(3, -\frac{1}{2})$.

Solution: We can use point-slope form, $y - y_1 = m(x - x_1)$, since $m = -\frac{2}{7}, (x_1, y_1) = \left(3, -\frac{1}{2}\right)$. So the equation of our line is $y - \left(-\frac{1}{2}\right) = -\frac{2}{7}(x - 3)$. We can simplify this to $y + \frac{1}{2} = -\frac{2}{7}(x - 3)$. We can also manipulate by isolating for y to re-write in slope-intercept form.

$$y = -\frac{2}{7}(x - 3) - \frac{1}{2}$$
$$\Rightarrow y = -\frac{2}{7}x + \frac{6}{7} - \frac{1}{2}$$
$$\Rightarrow y = -\frac{2}{7}x + \frac{6 * 2}{7 * 2} - \frac{1 * 7}{2 * 7}$$
$$\Rightarrow y = -\frac{2}{7}x + \frac{12}{14} - \frac{7}{14}$$
$$\Rightarrow y = -\frac{2}{7}x + \frac{5}{14}.$$

- Example 3: Write the equation of the line that goes through the points $(-1,4)$ and $(2, -1)$.

Solution: To use either point-slope or slope-intercept form, we need the slope. We can determine the slope by letting $(-1,4) = (x_1, y_1)$ and $(2, -1) = (x_2, y_2)$.
(Note it does not matter if we had picked $(-1,4) = (x_2, y_2)$, we would get the same result.)
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-4}{2-(-1)} = -\frac{5}{3}$. We can use point-slope since we already have two points, and we can use either point in the point-slope equation. Once simplified, both points will give us the same equation. Let's use $(-1,4)$ and our slope $m = -\frac{5}{3}$ in the point-slope equation.

$$y - 4 = -\frac{5}{3}\big(x - (-1)\big) \Rightarrow y - 4 = -\frac{5}{3}(x + 1)$$

We can manipulate into slope-intercept form by isolating for y on one side. So

$$y - 4 = -\frac{5}{3}(x + 1) \Rightarrow y - 4 = -\frac{5}{3}x - \frac{5}{3}$$
$$\Rightarrow y = -\frac{5}{3}x - \frac{5}{3} + 4 \text{ by adding four to both sides}$$

$\Rightarrow y = -\frac{5}{3}x - \frac{5}{3} + \frac{12}{3}$ turning 4 into a fraction to combine fractions

$$\Rightarrow y = -\frac{5}{3}x + \frac{7}{3}$$

Parallel or Perpendicular?

- Parallel lines:
  - What does it mean for two lines to be parallel?
    - The lines will never intersect each other.
  - How do we know mathematically that two lines will be parallel?
    - The slopes of the two lines will be equal each other, i.e. $m_1 = m_2$
      - Example: $y = 2x + 1,\ y = -\frac{1}{2} + 2x$

- Perpendicular lines:
  - What does it mean for two lines to be perpendicular?
    - They will intersect/cross at right angles at one point.
  - How do we know mathematically that two lines will be perpendicular?
    - The slopes of the two lines will be negative reciprocals of each other, i.e. $m_2 = -\frac{1}{m_1}$
      - Example: $y = 3x + 5\ ,\ y = -\frac{1}{3}x + 2$

Parallel or Perpendicular? Determine if the following lines are parallel, perpendicular, or neither:

- $y = \frac{2}{3}x - 4,\ \ y = -\frac{2}{3}x + 3$

Solution: The first line has slope of 2/3, while the second line has slope -2/3. These are not equal, nor negative reciprocals of each other. Thus the lines are neither parallel nor perpendicular.

- $y = -4,\ \ x = 1$. Parallel, perpendicular, or neither?

Solution: y=-4 is a horizontal line where all the y-values are -4. x=1 is a vertical line where all the x-values are 1. Horizontal and vertical lines are perpendicular to each other. We can draw and visually see these lines will be perpendicular as well.

- $y = \frac{1}{4}x + 5,\ \ 4y - x = -40$. Parallel, perpendicular, or neither?

Solution: The first line has slope ¼. The second line is in general form, so we can re-write the equation to find its slope. $4y - x = -40 \Rightarrow 4y = x + 40 \Rightarrow y = \frac{x}{4} + \frac{40}{4} \Rightarrow y = \frac{x}{4} + 10 \Rightarrow y = \frac{1}{4}x + 10$. Our slope for the second line is $\frac{1}{4}$. Since both slopes are ¼, the two lines are parallel.

- $2x - 3y = 12,\ 3x + 2y = 20.$ Parallel, perpendicular, or neither?

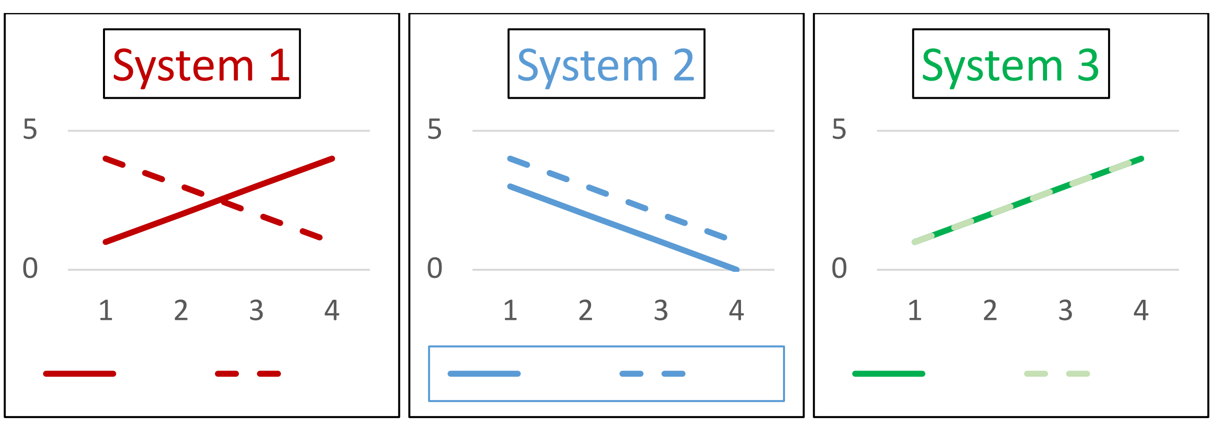
Solution: Both lines are in the general form. We can re-write both equations to help us determine their slopes more easily. $2x - 3y = 12 \Rightarrow -3y = 12 - 2x \Rightarrow y = \frac{12}{-3} - \frac{2x}{-3} \Rightarrow y = -4 + \frac{2}{3}x$. The slope of the first line is $\frac{2}{3}$. $3x + 2y = 20 \Rightarrow 2y = 20 - 3x \Rightarrow y = \frac{20}{2} - \frac{3}{2}x \Rightarrow y = 10 - \frac{3}{2}x$. The slope of the second line is $-\frac{3}{2}$. The two slopes are negative reciprocals of each other and thus perpendicular to each other.

Write one linear equation that is
a) parallel and
b) perpendicular to

$$y = -2x + 3$$

a) $y = -2x + 5$, or the equation of any line that is not $y = -2x + 3$ and has slope -2.
b) $y = \frac{1}{2}x + 5$, or the equation of any line that has slope $\frac{1}{2}$ since $\frac{1}{2}$ is the negative reciprocal of -2

Solving Systems



How many solutions can a system of two linear equations have?

System 1:

- One solution/point because the two lines will intersect/cross only once. How do we tell?
    - When you are solving, you will find only one x value and one y value.

System 2:

- No solutions because the two lines will never intersect/cross paths because they are parallel. How do we tell?
    - When you are solving, you will end up with an equation that will never be true/will not make sense

System 3:

- Infinitely many solutions because they are essentially the same line/overlap. How do we tell?
    - When you are solving, you will end up with an equation that will always be true.

How do we solve for the solutions of a system?
Method 1: Substitution.
Method 2: Elimination/Cancellation.

Find the solution(s) to the following system using **substitution**

$$\begin{cases} 2x - 4y = -5 \\ 3x + 2y = 8 \end{cases}$$

1. From one of the equations, we solve for one of the variables

$$2x - 4y = -5 \Rightarrow 2x = -5 + 4y \Rightarrow x = -\frac{5}{2} + \frac{4}{2}y \Rightarrow x = -\frac{5}{2} + 2y$$

2. Then we *substitute* the equation into the other equation to solve for the unknown variable.

$$3x + 2y = 8 \Rightarrow 3\left(-\frac{5}{2} + 2y\right) + 2y = 8 \Rightarrow -\frac{15}{2} + 6y + 2y = 8 \Rightarrow -\frac{15}{2} + 8y = 8 \Rightarrow 8y = 8 + \frac{15}{2} \Rightarrow 8y = \frac{16}{2} + \frac{15}{2} \Rightarrow 8y = \frac{31}{2}$$

$$\Rightarrow y = \frac{\frac{31}{2}}{8} \Rightarrow y = \frac{31}{2} * \frac{1}{8} \Rightarrow y = \frac{31}{16}$$

3. Then solve for the other variable by plugging into any of the equations. BUT the easiest one is the equation where the variable is already isolated.

$x = -\frac{5}{2} + 2y = -\frac{5}{2} + 2\left(\frac{31}{16}\right) = -\frac{5}{2} + \frac{31}{8} = -\frac{20}{8} + \frac{31}{8} = \frac{11}{8}$

Solution: one point, $\left(\frac{11}{8}, \frac{31}{16}\right)$

Find the solution(s) to the following system using **elimination/cancellation**

$$\begin{cases} 2x - 4y = -5 \\ 3x + 2y = 8 \end{cases}$$

1. Multiply one or both equations by values that will allow one of the variables to be eliminated when you combine/add the two equations together. This will give you one equation with one variable for which you would be able to solve for the variable.

$$\begin{cases} 2x - 4y = -5 \\ 2(3x + 2y = 8) \end{cases} \Longrightarrow \begin{cases} 2x - 4y = -5 \\ 6x + 4y = 16 \end{cases} \Longrightarrow 8x = 11 \Longrightarrow x = \frac{11}{8}$$

2. Then solve for the other variable by plugging into either of the equations.

$$3x + 2y = 8 \Longrightarrow 3\left(\frac{11}{8}\right) + 2y = 8 \Longrightarrow \frac{33}{8} + 2y = 8 \Longrightarrow 2y = 8 - \frac{33}{8}$$

$$\Longrightarrow 2y = \frac{64}{8} - \frac{33}{8} \Longrightarrow 2y = \frac{31}{8} \Longrightarrow y = \frac{\frac{31}{8}}{2} \Longrightarrow y = \frac{31}{8} * \frac{1}{2} \Longrightarrow y = \frac{31}{16}$$

Solution: one point, $\left(\frac{11}{8}, \frac{31}{16}\right)$. Observe this is the same solution we get when solving by substitution. Using both methods can help you check your work.

Practice: Find the solution(s) to $\begin{cases} 4x + 3y = 7 \\ 3x + 5y = 8 \end{cases}$

Using substitution with

Applications

We can come up with linear equations to help solve for what we are looking for, in math and with real world problems.

Example 1: You have a $50 gift card to your favorite takeout place and you want to order takeout for dinner. Tax is 10% on how much you spend and there is an additional $3 service fee. How much can you spend on food (for the subtotal) so that you do not have to pay out of pocket?

Solution: Let x be the amount that you can spend for the subtotal. Then $0.10x$ will be the amount that gets taxed on the subtotal. The total amount spent will be represented by $x + 0.10x + 3$, which equals 50. So $x + 0.10x + 3 = 50$. We can combine like terms and solve for x.

$1.1x + 3 = 50$. Then $1.1x = 47$, so $x = \frac{47}{1.1} = 42.727272$ … Which means you would be able to spend a subtotal of $42.72 on food without having to pay out of pocket. Note that we have to round down, otherwise if we spend a subtotal of $42.73 or more on food, then we will have to pay out of pocket since the total will go over the $50 gift card.

Example 2: You want to build a fence around the yard to keep the dog in and have 5000 yards of material left to build a rectangular perimeter. You want the length to be 100 yards longer than twice the width. How long should the length and width of the perimeter of the fencing be?

Solution: Let L be the length of the rectangular perimeter and W be the width of the rectangular perimeter. We know that adding up all the sides of the rectangle gives the entire perimeter, which will equal 5000. So L+W+L+W=5000. Simplified this gives 2L+2W=5000. Since the length is 100 yards longer than twice the width, mathematically this translates to the equation L=100+2W. We can use substitution in the perimeter equation and we get

2L+2W=2(100+2W)+2W=5000. So 200+4W+2W=5000. Simplifying the left hand side we get 200+6W=5000. We can solve for W by isolating for W. 6W=5000-200 => 6W=4800. Divide both sides by 6 to get W=800. If the width W is 800 yards, then length L will be 100+2W=100+2(800)=1700 yards.

Absolute Value Equations

- Recall that the absolute value of a number is its distance from 0 on the number line. Absolute values are always greater than or equal to 0. Examples: $|-11| = 11,\ |11| = 11$.
- For any algebraic expression, x, and any positive real number, n, if $|x| = n$, then $x = n$ or $x = -n$.
- How do we solve absolute value equations?
    - Step 1: Isolate the absolute value expression
    - Step 2: Write the equivalent equations. I.e. set the inside expression equal to the positive value it is equal to, and set the inside expression equal to the negative value
    - Step 3: Solve each equation
    - Step 4: Check each solution
- Solve $|3x - 4| = 2$

The two equations this absolute value equation is equivalent to are
$3x - 4 = 2$ and $3x - 4 = -2$

$3x - 4 = 2$	We want to solve for x by:	$3x - 4 = -2$
$3x = 6$	Adding 4 to both sides to each equation	$3x = 2$
$x = \frac{6}{3}$	Dividing both sides by 3 to each equation	$x = \frac{2}{3}$
$x = 2$	Simplify answers	$x = \frac{2}{3}$
$|3(2) - 4| = 2?$ $|6 - 4| = 2$ $2 = 2$ This is a solution	Check answers	$\left|3\left(\frac{2}{3}\right) - 4\right| = 2?$ $|2 - 4| = 2$ $|-2| = 2$ $2 = 2$ This is a solution

- Solve $4|x - 1| + 2 = 10$

Solution: Isolate for the absolute value first. Subtract 2 from both sides and we get $4|x - 1| = 8$. Then divide both sides by 4 to get $|x - 1| = 2$. Now we can write the two equations the absolute value equation is equivalent to. $x - 1 = 2$ and $x - 1 = -2$. So we will add 1 to both sides of the two equations and get $x = 3$ and $x = -1$. Check if the solutions are correct:
$4|x - 1| + 2 = 4|3 - 1| + 2 = 4|2| + 2 = 4(2) + 2 = 8 + 2 = 10$, so yes $x = 3$ is a solution.
$4|-1 - 1| + 2 = 4|-2| + 2 = 4(2) + 2 = 8 + 2 = 10$, so yes $x = -1$ is a solution.

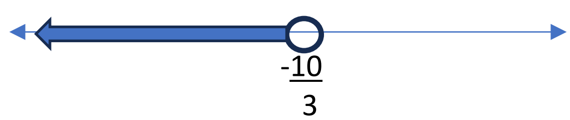
Linear Inequalities

- A linear inequality is an inequality in one variable that can be written in one of the following forms where a, b, c are real numbers and $a \neq 0$. $ax + b < c,\ ax + b \leq c,\ ax + b > c, ax + b \geq c$
- Solve linear inequalities the way you would a regular linear equation. The difference is we have an inequality symbol instead of = and the solution could be a set of values.
- Properties:
  - Addition/subtraction: whatever you do to one side, do to the other
  - Multiplication/division: whatever you do to one side, do to the other BUT
    - Gotta be careful though! If you multiply or divide by a negative value, we have to flip the symbol of the inequality. Example: 3<4. If we multiply by -1, do we get -3<-4? This is not true, -3>-4. This is why we have to flip the symbol.
- Writing in interval notation:
  - Parenthesis will not include the value
  - Bracket includes the value
  - ∞ or $-\infty$ cannot be included since it is not a finite value that can be reached

In the following exercises, solve, graph, and write solution in interval notation.

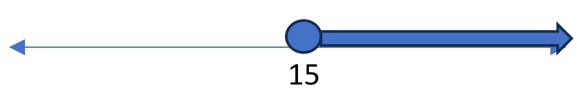
- $6p > 15p - 30$

Solution: Like solving a linear equation, we want to isolate for the variable. We start by subtracting 15p from both sides
$-9p > -30$ then divide both sides by -9. Since we are dividing by a negative, we will want to flip the inequality symbol and we will get $p < -\frac{30}{9}$. We can reduce the fraction to $-\frac{10}{3}$. So $p < -\frac{10}{3}$. These are all the values that are less than $-\frac{10}{3}$. Thus our interval notation would be $(-\infty, -\frac{10}{3})$. Graphing would have a hollow circle around $-\frac{10}{3}$ and shaded everything to the left of $-\frac{10}{3}$.



- $9h - 7(h - 1) \leq 4h - 23$

Solution: Before we can start isolating for the variable, we must distribute the -7.
$9h - 7h + 7 \leq 4h - 23$. Combine like terms $2h + 7 \leq 4h - 23$. Add 23 to both sides to get $2h + 30 \leq 4h$. Subtract 2h from both sides. $30 \leq 2h$. Divide both sides by 2. $15 \leq h$. All the values are greater than or equal to 15. We can rewrite $15 \leq h$ as $h \geq 15$. As an interval, the inequality would be represented as $[15, \infty)$. Graphically, there would be a filled in circle at 15, and everything shaded to the right of 15.
Note: you could have started by subtracting 7, or or either of the terms with the variable from both sides. You would still end up with the same answer once simplified.



Absolute Value Inequalities

- For any algebraic expression, x, and any positive real number, n,
  - If $|x| < n$, then $-n < x < n$
  - If $|x| \leq n$, then $-n \leq x \leq n$
- How do we solve absolute value inequalities with $<$ or $\leq$?
  - Step 1: Isolate the absolute value expression
  - Step 2: write the equivalent compound inequality
  - Step 3: Solve the compound inequality
  - Step 4: Graph the solution
  - Step 5: Write the solution using interval notation

In the following exercises, solve, graph, and write solution in interval notation.

- $|2x - 5| \leq 3$

Solution: We start from step 2 since the absolute value expression is already isolated on one side. The equivalent compound inequality is
$-3 \leq 2x - 5 \leq 3$
$2 \leq 2x \leq 8$ by adding 5 to each part of the inequality
$1 \leq x \leq 4$ by divide each part by 2
Visually we draw a number line and put filled in circles at both 1 and 4, and shade everything between 1 and 4 because the inequality shows x will be any number between 1 and 4, including 1 and 4. In interval notation this is represented as $[1,4]$.

- $|5x + 1| < 4$

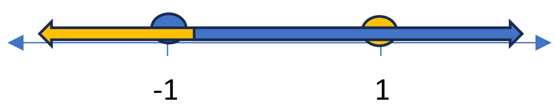
Solution: We start from step 2 since the absolute value expression is already isolated on one side. The equivalent compound inequality is
$-4 < 5x + 1 < 4$
$-5 < 5x < 3$ by subtracting 1 to each part of the inequality
$-1 < x < \frac{3}{5}$ by divide each part by 2
Visually we draw a number line and put hollow circles at both -1 and 3/5, and shade everything between -1 and 3/5. In interval notation this is represented as $(-1, \frac{3}{5})$ since -1 and 3/5 are not included in the solution set.

- For any algebraic expression, x, and any positive real number, n,
  - If $|x| > n$, then $x > n$ or $x < -n$
  - If $|x| \geq n$, then $x \geq n$ or $x \leq -n$
- How do we solve absolute value inequalities with $>$ or $\geq$?
  - Step 1: Isolate the absolute value expression
  - Step 2: write the equivalent compound inequality
  - Step 3: Solve the compound inequality
  - Step 4: Graph the solution
  - Step 5: Write the solution using interval notation

In the following exercises, solve, graph, and write solution in interval notation.

- $3|x| + 4 \geq 1$

Solution: We start by isolating for the absolute value expression. Subtract 4 from both sides. Then $3|x| \geq -3$. Then divide both sides by 3 to get $|x| \geq -1$. The compound inequalities are $x \geq -1$ or $x \leq 1$. Visually the first inequality is everything greater than or equal to -1 (shade -1 and everything to the right of -1). The second inequality is everything less than or equal to 1 (shade 1 and everything to the left of 1). Since this is a compound inequality that is an "or" statement, we include solutions from both inequalities. Which means the solution set is all real numbers and written in interval notation is $(-\infty, \infty)$.



- $|x - 5| > -2$

Solution: We start at step 2 since the absolute value expression is already isolated for. So the compound inequalities we want to solve are $x - 5 > -2$ or $x - 5 < 2$. We add 5 to both sides of each inequality and for the first one we get $x > 3$ and for the second inequality we get $x < 7$. Since this is compound inequalities with an "or" statement, we again include the solutions from both inequalities, which leaves us with all the real numbers and written in interval notation is $(-\infty, \infty)$.

Workshop Survey:

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